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Downtime Distributions Based on a Multivariate Exponential Distribution

by
C.L.Hsu
L.Shaw

Program in
System and Device Reliability

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Sums of dependent exponentially distributed random variables are examined as models for system downtimes. A multivariate exponential distribution derived by representing each exponential variable as a sum of squares of two normal variables was found to be very convenient. It was found that any distribution which applies to a sum of n such dependent exponential variables can also be achieved by summing n-independent exponential variables. The latter kinds of sums are known to give reasonable approximations to lognormal variables.

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1. Introduction

The design of maintenance policies for maintainable and repairable systems makes use of information about probability distributions of component lifetimes and downtimes. Since such designs are facilitated if these distributions have simple analytic forms, downtime distributions have been frequently modeled as lognormal, Weibull or Erlarg in form(1). These distributions all are skewed and correspond to non-negative random variables like downtimes. Here we consider one more family of distributions which has some physical motivation.

Since a downtime interval is often the sum of subsidiary intervals (for failure isolation, component removal, repair, reassembly, alignment, etc.) it seems reasonable to think of the downtime x_n as a sum of subsidiary time intervals

$$x_n = \sum_{i=1}^{n} r_i \tag{1.1}$$

The subscribt on x_n reminds us of the number of summands. Several distributions are possible for the individial r_i, but here we consider exponential distributions which are the simplest and which are widely used to represent random times between events.

It is well known that the down-time x_n will have an Erlarg distribution if the r_i are independent exponential variables with identical mean values. Muth [2] has considered the approximation of Weibull and lognormal distributions by x_n in which the r_i are independent exponential variables but with possibly different mean values. Here we further generalize to allow dependence among the r_i------a reasonable situation if the vairables represent related steps in a sequence of downtime operations.

Section 2 discusses several forms of multivariate distributions which have exponential mariginal distributions. The one based on the sums of sequences of normal variables is selected, because of its analytical simplicity, for use in the Section 3 calculations of downtime distributions. In addition to giving examples of these correlated-sums distributions. We show that introduction of dependence among r_i, (with possibly unequal means) does not broaden the class of x_n distributions over that which results from independant r_i. That is, the sum of n dependent exponential variables has a distribution identical to that of the sum of n other independent exponential variables.

Concluding comments refer to approximation of lognormal variables by sums of exponential ones, along with other possible extensions.

2. Multivariate Exponential Distributions

It is well known that a multivariate distribution is not uniquely specified by its marginal distributions(5). For example, while the bivariate normal distribution has nice analytical properties, it is not the only one with normal marginals. This multiplicity will be demonstrated for the case of exponential marginals by considering a few possible bivariate densities.

Gumbel [3] considered several bivariate exponential densities. The first $F_i(r_1, r_2)$ is based on the following general formula for combining marginal distributions:

$$F(x, y) = F_x(x)\, F_y(y) \left\{ 1 + \alpha\, [\, 1-F_x(x)\,]\, [\, 1-F_y(y)\,] \right\} \qquad (2.1)$$

$$|\alpha| \leq 1$$

When applied to exponential variables with unit mean values this produces

the distribution function

$$F_1(r_1, r_2) = (1-e^{-r_1})(1-e^{-r_2})(1+\alpha e^{-r_1-r_2}) \tag{2.2}$$

$$r_1, r_2 \geq 0$$

$$|\alpha| \leq 1$$

and the density function

$$f_1(r_1, r_2) = e^{-r_1-r_2}\left[1+\alpha(2e^{-r_1}-1)(2e^{-r_2}-1)\right] \tag{2.3}$$

In this model, $\alpha = 0$ corresponds to independence of r_1 and r_2, and it can be shown that the correlation coefficient ρ_{r_1, r_2} is

$$\rho_{r_1, r_2} = \alpha/4 \tag{2.4}$$

with its magnitude limited to be less than 1/4.

Another model of Gumbel's is defined by the distribution function

$$F_2(r_1, r_2) = 1-e^{-r_1}-e^{-r_2}+e^{-r_1-r_2-\theta r_1 r_2} \tag{2.5}$$

$$r_1, r_2 \geq 0$$

$$0 \leq \theta \leq 1$$

and the corresponding density

$$f_2(r_1, r_2) = e^{-r_1-r_2-\theta r_1 r_2}\left[(1+\theta r_1)(1+\theta r_2) - \theta\right] \tag{2.6}$$

Here $\theta = 0$ corresponds to independence, and

$$\rho_{r_1, r_2} = -\theta^{-1} e^{+1/\theta} \, \mathrm{Ei}\,(\theta^{-1}) - 1$$

(Ei is an exponential integral.)

Marshall and Olkin [4] introduced the bivariate exponential defined by

$$\bar{F}\,(r_1, r_2) = P\left[R_1 > r_1, R_2 > r_2\right]$$

$$= e^{-\lambda_1 r_1 - \lambda_2 r_2 - \lambda_{12} \max(r_1, r_2)} \tag{2.7}$$

or

$$F_3(r_1, r_2) = 1 - e^{-(\lambda_1 + \lambda_{12}) r_1} - e^{-r_2(\lambda_2 + \lambda_{12})}$$

$$+ e^{-r_1 \lambda_1 - r_2 \lambda_2 - \lambda_{12} \max(r_1, r_2)} \tag{2.8}$$

Here the correlation coefficient is

$$\rho_{r_1, r_2} = \lambda_{12}/(\lambda_1 + \lambda_2 + \lambda_{12}) \tag{2.9}$$

which ranges between zero and one. This distribution can be thought of as a result of fatal shocks occuring from three independent Poisson sources with rates λ_1, λ_2 and λ_{12}. Component 1 with lifetime r_1 is killed by events of the first or third variety, and r_2 is determined by events of the second or third type. Barlow and Prochan [5] point out that (2.8) is the unique exponential bivariate with the zero - memory property: The joint

survival probability of a pair of components each of age t is the same for all t (e.g. the same as if both were new). This zero memory property is quite desirable when modeling joint lifetimes of components in a system. In the present context of r_i representing durations of a sequence of related events, this special property seems inessential.

Kibble[6] considered a bivariate exponential density of the form

$$F_4(r_1, r_2; \rho) = \frac{1}{4\sigma^4(1-\rho^2)} \left[e^{-\frac{r_1+r_2}{2\sigma^2(1-\rho^2)}} \right] I_0 \left[\frac{\sqrt{\rho^2 r_1 r_2}}{\sigma^2(1-\rho^2)} \right] \quad (2.10)$$

in which I_0 is a modified Bessel function .

We will derive this in an alternate, more natural way for use below. The density in (2.10) applies when we view r_1 and r_2 as being generated from correlated normal variables. It is well known that if w_1 and z_1 are independent, zero mean, equal variance (σ^2) normal variables then r_1 defined as

$$r_1 = w_1^2 + z_1^2 \quad (2.11)$$

has an exponential distribution with mean

$$E(r_i) = 2\sigma^2 \quad (2.12)$$

Now if (w_1, w_2) and (z_1, z_2) are two independent pairs of normal variables, but with

$$\operatorname{cov}(w_i, w_j) = \operatorname{cov}(z_i, z_j) \quad \begin{cases} i = 1,2 \\ j = 1,2 \end{cases} \quad (2.13)$$

then r_1 and r_2 defined by

$$r_i = w_i^2 + z_i^2 \qquad i = 1, 2 \tag{2.14}$$

will be dependent exponential variables.

Equation (2.10) can be derived from this reasoning when all four variables (w_1, w_2, z_1, z_2) have equal variances σ^2. In that case

$$f(w_1, w_2, z_1, z_2) = \frac{1}{(2\pi)^2(1-\rho^2)\,\sigma^4} \exp\left[- \frac{w_1^2 - 2\rho w_1 w_2 + w_2^2 + z_1^2 - 2\rho z_1 z_2 + z_2^2}{2(1-\rho^2)\,\sigma^2} \right] \tag{2.15}$$

and

$$F_4(r_1, r_2) = \iiiint f(w_1, w_2, z_1, z_2)\, dw_1, dw_2, dz_1, dz_2$$

$$\left\{ \begin{array}{l} w_1^2 + z_1^2 \leq r_1 \\ w_2^2 + z_2^2 \leq r_2 \end{array} \right\} \tag{2.16}$$

Introduction of polar coordinates in w_1, z_1 and w_2, z_2 planes

$$w_1 = \gamma_1 \cos\theta_1 \qquad w_2 = \gamma_2 \cos\theta_2$$

$$z_1 = \gamma_1 \sin\theta_1 \qquad z_2 = \gamma_2 \sin\theta_2$$

reduce the F_4 integral to the form

$$F_4(r_1, r_2) = \frac{1}{4\pi^2(1-\rho^2)\,\sigma^4} \int_{\gamma_1=0}^{\sqrt{r_1}} \int_{\gamma_2=0}^{\sqrt{r_2}} \int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{2\pi}$$

$$\bullet\; e^{-\left[\gamma_1^2+\gamma_2^2 - 2\rho\,\gamma_1\,\gamma_2 \cos(\theta_1-\theta_2)\;/\;2\,\sigma^2\,(1-\rho^2)\right]} \tag{2.17}$$

$$\bullet\; \gamma_1\,\gamma_2\, d\gamma_1\, d\gamma_2\, d\theta_1\, d\theta_2$$

Substitution of $\varphi = \theta_1 - \theta_2$ for the θ_1 integration produces a periodic integrand, independent of θ_2. Thus the θ - integrals become

$$\int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{2\pi} = 2\pi \int_0^{2\pi} e^{\rho\,\gamma_1\,\gamma_2 \cos\varphi\,/\,\sigma^2(1-\rho^2)}\, d\varphi$$

$$= 4\pi^2 \quad I_0\left[\frac{|\rho|\;\gamma_1\,\gamma_2}{\sigma^2(1-\rho^2)}\right] \tag{2.18}$$

Finally substitution of (2.18) into (2.17) and differentiations with respect to r_1 and r_2 produces (2.10). It turns out that ρ_{r_1, r_2} is just the square ρ^2 of the correlation of the underlying normal variables, with a range between zero and one.

The Bessel function form for this bivariate exponential based on normal variates may not appear to be very felicitous. However, this kind of distribution will be convenient when we concentrate in the next section on the sum of dependent exponential variables, as in (1.1).

For the n- dimensional version of this class of distributions, we consider zero mean, normal n-vectors $\underline{w}$ and $\underline{z}$ each with the same covari-

ance matrix

$$E[\underline{w}\ \underline{w}'] = E[\underline{z}\ \underline{z}'] = \Gamma \qquad (2.19)$$

In this way, for each i w_i and z_i will have the same variances so the sum of their squares will be an exponential random variable r_i. We do allow r_i and r_j to have unequal means, contrary to the special case in (2.10).

3. Examples and Properties of Proposed Family of Distributions

The previous sections introduced the multivariate exponential distribution based on a pair of multivariate normal distrubtions. General expressions for this distribution were not presented, but we are only interested in the distribution of the sum of the dependent exponential variables.

Some general ideas about such distributions are revealed in the following simple examples. We consider

$$x_2 = r_1 + r_2 \qquad (3.1)$$

the sum of two equal mean exponentials which have the joint density in (2.10). As we will see in a more general context below, the characteristic function of x_2 is

$$\Phi_2(s) = E[\ e^{-s\,x_2}\ ]$$

$$= \frac{1}{4(1-\rho^2)\,s^2 + 4\,s + 1} \qquad (3.2)$$

when $E[\,r_i\,] = E\,[\,r_2\,] = 2\,\sigma^2 = 2$. Table I shows several examples of f_{x_2} as ρ varies between 0 and 1. (Recall the ρ is the correlation coefficent of the underlying normal variables and ρ^2 is the correlation coefficient of

r_1 and r_2). In each case $E[x_2] = 4$, and the peak of the density moves left toward the limiting ($\rho = 1$) exponential case as ρ increases.

Table I reveals another interesting fact. This is that for each value of ρ relating r_1 and r_2 we can always find two independent exponential variables r'_1 and r'_2 with possibly different mean values, such that $x'_2 = r'_1 + r'_2$ has the same distribution as $x_2 = r_1 + r_2$.

Another interesting comparison is that of $x_2, x_3, \ldots$ based on addition of more and more dependent exponential random variables. The natural extension of the bivariate example in (3.2) is to consider all underlying normal variables to have variances of σ^2, and with the corresponding $\underline{w}$ and $\underline{z}$ vectors each having the covariance matrix

$$\Gamma_n = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \rho^2 \cdots \\ \vdots & & \ddots & & \\ \rho^{n-1} & & & & 1 \end{pmatrix} \sigma^2 \tag{3.3}$$

This implies that the r_i have the correlation coefficient

$$\rho_{r_i r_j} = \rho^{2|i-j|} \tag{3.4}$$

If we further consider mormalized variables

$$\hat{x}_n = \frac{1}{2n\sigma^2} \sum_1^n r_i \tag{3.5}$$

then

$$E[\hat{x}_n] = 1 \tag{3.6}$$

Calculation of other moments for these $\hat{x}_n$ is straightforward, though tedious, from the moment generating properties of the underlying normal distributions. Moment expressions for the special example using (3.3), given in the appendix, show the following limitations on the possible values of variance σ_n^2, Skewness ζ_n, and Kurtosis ν_n.

$$\begin{aligned} \frac{1}{n} &\leq \sigma_n^2 \leq 1 \\ 2\sigma_n &\leq \zeta_n < 2 \\ 6\sigma_n^2 &\leq \nu_n < 6 \end{aligned} \tag{3.7}$$

Figure 1 shows the effects on the density function shape of increasing the number of equal mean exponential variable which are summed. Those curves are for $\hat{x}_2$ and $\hat{x}_3$ each with unit mean and variance of $\sigma_2^2 = \sigma_3^2 = 0.65364583$ (corresponding respectively to $\rho = 0.55433895$ and $\rho = 3/4$). The equations for those densities were found to be

$$f_{x_2}(t) = 1.8039505\left(e^{-1.2867206t} - e^{-4.4877156t}\right)$$

$$f_{x_3}(t) = 0.99159025e^{-16.310164t} + \tag{3.8}$$

$$+ \; 1.6750764 \, e^{-1.2612644t}$$

$$- \; 2.666667 \, e^{-6.8571428t}$$

The similarity of the two curves in the figure attests to the need for great precision in expressing the densities and in computing typical points.

Figure 1 also shows the lognormal density with the same mean and variance as $\hat{x}_2$ and $\hat{x}_3$.

The relation between families of distributions which was suggested by the results in Table I will now be generalized in the form of a Theorem. We define $r_1, r_2, \ldots r_n$ as sums

$$r_i = w_i^2 + z_i^2 \tag{3.9}$$

in terms of two independent normal n-vectors $\underline{w}$ and $\underline{z}$, each with zero mean values and covariance matrix Γ. As shown above, the r_i are exponentially distributed with mean

$$E[\, r_i \,] = 2 \, \gamma_{ii} \tag{3.10}$$

and correlation coefficients

$$\rho_{r_i r_j} = (\gamma_{ij})^2 \tag{3.11}$$

<u>Theorem: All possible density functions</u> $f_{x_n}(t)$ <u>for</u> $x_n = \sum_1^n r_i$ <u>can be achieved with independent</u> r_i, <u>i.e. with diagonal</u> $\Gamma = \mathrm{diag}(\alpha_1, \alpha_2 \ldots \alpha_n)$.

This Theorem is proved by working with characteristic functions, as follows. Let

$$v = \sum_1^n w_i^2; \quad y = \sum_1^n z_i^2 \tag{3.12}$$

then

$$\Phi_{xn}(s) = \Phi_v(s)\,\Phi_y(s) = \Phi_v^2(s) \tag{3.13}$$

due to the independence and identical distributions of $\underline{w}$ and $\underline{z}$. The possibly correlated variables w_i can be represented as linear transformations of independent unit variance normal variables ζ_i:

$$\underline{w} = M\underline{\zeta}, \qquad E[\zeta\,\zeta'] = I \tag{3.14}$$

similarly

$$\underline{z} = M\underline{\xi} \tag{3.15}$$

it follows that

$$v = \sum_1^n w_i^2 = \underline{\zeta}'M'M\,\underline{\zeta} \tag{3.16}$$

and

$$\Phi_v(s) = \left(\sqrt{2\pi}\right)^{-n} \int \cdots \int \exp\left[-\frac{1}{2}\,\underline{\zeta}'\underline{\zeta} - s\,\underline{\zeta}'\,M'M\,\underline{\zeta}\right] d\underline{\zeta}$$

$$= (\sqrt{2\pi})^{-n} \int \ldots \int \exp\left[-\frac{1}{2} \underline{\zeta}' R^{-1} \underline{\zeta}\right] d\,\underline{\zeta} \tag{3.17}$$

where we have defined the matrix

$$R^{-1} = I + 2\,s\,M'M \tag{3.18}$$

The integral in (3.17) is of the form of a normal density integrated over all values, except, for a scale factor.

Thus

$$\Phi_v(s) = \left(\sqrt{|R^{-1}|}\right)^{-1} \tag{3.19}$$

and the desired characteristic function for the sum of exponentical variables is

$$\Phi_{x_n}(s) = \left(|I + 2\,s\,M'M|\right)^{-1} = 1/q(s) \tag{3.20}$$

(several examples are found in the appendix).

The roots of q(s) are negative reciprocals of the eignvalues of the real symmetric matrix 2 M' M, so they are real numbers. If the r_i are independent then M is diagonal

$$M_I = \text{diag}\,(\alpha_1, \alpha_2 \cdots \alpha_n) \tag{3.21}$$

and so is

$$2\,M_I{}'\,M_I = \text{diag}\,(2\,\alpha_1^2, 2\alpha_2^2, \ldots 2\alpha_n^2) \tag{3.22}$$

In this " independent " case

$$q_I(s) = \prod_{i=1}^{n} (1 + 2\,\alpha_i^2\, s) \qquad (3.23)$$

The proof is completed by comparing the polynomials q(s) in (3. 20), for a general M, to the $q_I(s)$ in (3.23) for independent r_i. The mean values of the latter ($2\,\alpha_i^2$) can be chosen to make the roots of $q_I(s)$ match any n (necessarily real) roots of q(s). The resulting polynormals and characteristic functions will be identical because the $\Phi(0) = 1$ property removes any scale factor ambiguity. This completes the proof.

In general, the n-independent exponential variables whose sum is indistinguishable from the sum of n- correlated ones will have different mean values from those of the correlated variables.

Once this structure has been established for sums of correlated exponential variables, previous results for sums of independent variables (e.g. those of Muth[2]) are directly applicable. However, it appears that when equal mean variables have their mean and correlation ρ in (3. 3) adjusted so x_n matches the lognormal mean and variance, then f_{x_n} tends to lognormal for large n. This is in contrast to Muth's summing of independent variables in which he had to search for the proper individual mean values which made the sum approximately lognormal.

Other interesting properties follow from knowledge of the general form of the solution. For example, if all roots of (11) are distinct then

$$f_{x_n}(t) = \sum_{i=1}^{n} a_i e^{-\lambda_i t} \qquad (3.24)$$

(The roots must be negative since f(t) must have finite area). One condition on the a_i is that

$$\int_0^{\infty} f_{x_n}(t)\,dt = 1 = \sum_{1}^{n} \frac{a_i}{\lambda_i} \qquad (3.25)$$

application of the initial value Theorem of Laplace transforms to (3.20) shows further that

$$f_{x_n}^{(\kappa)}(0) = 0 \quad \kappa = 0, 1, \ldots (n-2) \tag{3.26}$$

$$= \sum_{1}^{n} (-\lambda_i)^{\kappa} a_i$$

Equations (3.25) and (3.26) provide n linear equations which can be solved for the n coefficients a_i in (3.24), if the roots λ_i are known.

4. Comparisons with Lognormal Distribution

A lognormal random variables x is defined by saying that its natural logarithm has a normal distribution with mean η and variance σ^2. Thus, the density of x is

$$f(x) = (x\sigma\sqrt{2\pi})^{-1} e^{-(\ell n\, x - \eta)^2/2\sigma^2} \quad ; \; x \geq 0 \tag{4.1}$$

and its mean and variance can be expressed as

$$E[x] = e^{\eta + \sigma^2/2} \tag{4.2}$$

$$Var[x] = e^{2\eta + \sigma^2}[e^{\sigma^2} - 1] \tag{4.3}$$

$$= E^2[x][e^{\sigma^2} - 1]$$

Figure 1 shows a typical lognormal density, which is necessarily defined for positive x and is unimodal.

Lognormal models have been used frequently to characterize down-time data, probably because of the ease in doing so with the aid of normal probability paper. Many other simple unimodal distributions can often be used with equally good effect with a finite amount of data.

One special property of the lognormal is its behavior for large arguments. One way to describe this behavior is in terms of the conditional mean exceedance

$$CME_{\tau} = E[\ t - \tau \mid t > \tau\ ] = \int_{\tau}^{\infty} \bar{F}(t)\, dt \,/\, \bar{F}(\tau) \qquad (4.4)$$

If t were a lifetime random variable, then CME_{τ} would be the expected future life as seen at time τ. CME_{τ} is a constant for an exponential random variable, but is an increasing function of τ, when τ is large, for a lognormal random variable. Thus, the latter is said to have a heavy tail.

It is clear that the tail of an x_n as defined in the previous sections will not be heavy, since each density there has the form of a sum of decaying exponentials. The tail of such a density will be dominated by the slowest of these exponentials, so it will be neither heavy nor light.

5. Conclusions

Several multivariate exponential distributions were examined for use in modeling constituent time intervals whose sum represents a system's downtime. The distribution based on sums of squares of pairs of normal variables was chosen for its analytic simplicity.

We showed that variation in the correlation and means of such variables does not produce any distributions for the sum which could not have been realized by a sum of independent exponential variables. Such sums can be found to well approximate the frequently used lognormal downtime distribution, except for the tail behavior. It is not clear that the heavy tailed behavior of the lognormal is truly more appropriate for down time modeling. A lognormal model could be justified for a variable resulting from a product of independent variables (by applying the Central Limit Theorem to its logarithm), but there seems to be no physical justification for such

a downtime model.

The multivariate exponential model studied here may also be useful in generalizing maintenance and replacement strategy modeling by introducing dependent intervals into problems where independent variables have been used previously. It is interesting, in this context, to note that when a development paralleling that of (2.10) is applied to sets of three correlated normal variables with

$$E[w_i w_j] = \sigma_i \sigma_j \rho^{|i-j|}$$

then the trivairate exponential density becomes

$$f(r_1, r_2, r_3) = \frac{1}{8\sigma_1^2 \sigma_2^2 \sigma_3^2 (1-\rho^2)} e^{-\left[\left(\frac{r_1}{\sigma_1^2} + \frac{r_2(1+\rho^2)}{\sigma_2^2} + \frac{r_3}{\sigma_3^2}\right) / 2(1-\rho^2)\right]}$$

$$\cdot I_0\left[\frac{\sqrt{r_1 r_2 \rho^2}}{\sigma_1 \sigma_2 (1-\rho^2)}\right] I_0\left[\frac{\sqrt{r_2 r_3 \rho^2}}{\sigma_2 \sigma_3 (1-\rho^2)}\right]$$

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APPENDIX

Moments for Sums of Dependent Exponential Random Variables:

Case of Equal Means and Exponential Correlation Coefficient

This appendix gives further details for the distributions introduced in Section 3 and defined by (3.3) with normalization (3.5). We consider cases for n = 2, 3, and 4 using (3.20) get the characteristic functions

$$\Phi_2(s) = [\,0.25(1-\rho^2)\,s^2 + s + 1]^{-1}$$

$$\Phi_3(s) = [\,(8-16\rho^2 + 8\rho^4)\,(s/6)^3 + (12-8\rho^2 - 4\,\rho^4)\,(s/6)^2 + s + 1\,]^{-1}$$

$$\Phi_4(s) = [(16-48\rho^2 + 48\rho^4 - 16\rho^6)\,(s/8)^4 + (32 - 48\rho^2 + 16\rho^6)\,(s/8)^3$$

$$+ (24-12\rho^2 - 8\rho^4 - 4\rho^6)\,(s/8)^2 + s + 1\,]^{-1}$$

Moments of interest are defined as follows:

$$m_\kappa(n) = E\,[\,x_n^{\,\kappa}\,]$$

$$\text{mean} = \mu = m_1(n)$$

$$\text{Variance} = \sigma_n^2 = m_2(n) - m_1^2\,(n)$$

$$\text{Shewness} = \zeta_n = E\left[\left(\frac{x_n - m_1(n)}{\sigma_n}\right)^3\right]$$

$$\text{Kurtosis} = \nu_n = E\left[\left(\frac{x_n - m_1(n)}{\sigma_n}\right)^4\right] - 3$$

$$= \sigma_n^{-3} \left[m_3(n) - 3\, m_1(n)\, m_2(n) + 2\, m_1^{\,3}(n) \right]$$

$$= \sigma_n^{-4} \left[m_4(n) - 4\, m_1(n)\, m_4(n) + 6\; m_1^{\,2}(n)\, m_2(n) - 3\, m_1^{\,4}(n) \right] - 3$$

Formulas for these moments as functions of ρ are given below:

$n = 2$

$$\mu = 1$$

$$\sigma^2 = \frac{1}{2} + \frac{1}{2}\ \rho^2$$

$$\zeta = \frac{1}{\left(\frac{1}{2} + \frac{1}{2}\ \rho^2\right)^{3/2}} \ (0.5 + 1.5\ \rho^2)$$

$$\nu = \frac{1}{(0.5 + 0.5\ \rho^2)^2} \ (1.5 + 6\rho^2 + 1.5\ \rho^4)$$

$n=3$

$$\mu = 1$$

$$\sigma^2 = \frac{1}{3} + \frac{4}{9}\ \rho^2 + \frac{2}{9}\ \rho^4$$

$$\zeta = \frac{1}{\left(\frac{1}{3} + \frac{4}{9}\ \rho^2 + \frac{2}{9}\ \rho^4\right)^{3/2}} \ (0.222222 + 0.888888\ \rho^2 + 0.888888\ \rho^4)$$

$$\nu = \frac{1}{(\sigma^2)^2} \ (0.55555 + 2.66666\ \rho^2 + 3.501673\ \rho^4 + 1.185185\ \rho^6 + 0.296296\ \rho^8) - 3$$

$n = 4 \qquad \mu = 1$

$$\sigma^2 = 0.25 + 0.375\ \rho^2 + 0.25\ \rho^4 + 0.125\ \rho^6$$

$$\zeta = \frac{1}{(\sigma^2)^{3/2}} (0.125 + 0.5625\ \rho^2 + 0.75\ \rho^4 + 0.375\ \rho^6)$$

$$\nu = \frac{1}{(\sigma^2)^2} (0.28125 + 1.40625\ \rho^2 + 2.8125\ \rho^4 - 0.28125\ \rho^6 +$$

$$0.7529297\ \rho^8 + 0.375\ \rho^{10} + 0.09375\ \rho^{12}) - 3$$

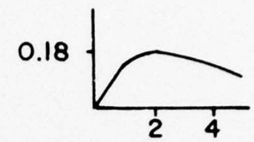
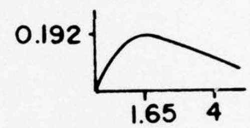
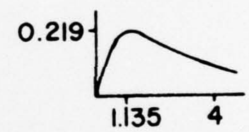
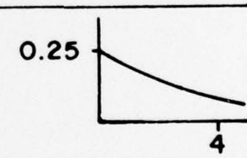
	(a)	(b)		
$E[r_1] = E[r_2] = 2$		$\rho = 0$		
i)	$\rho = 0$	$E[\hat{r}_1] = 2$ $E[\hat{r}_2] = 2$	$f_{\underset{\sim}{x}_2}(x_2) = \frac{1}{4}\, x_2 e^{-0.5 x_2}$	
ii)	$\rho = \frac{1}{4}$	$E[\hat{r}_1] = 2.5$ $E[\hat{r}_2] = 1.5$	$f_{\underset{\sim}{x}_2}(x_2) = e^{-0.4x_2} - e^{-\frac{2}{3}x_2}$	
iii)	$\rho = \frac{1}{2}$	$E[\hat{r}_1] = 3$ $E[\hat{r}_2] = 1$	$f_{\underset{\sim}{x}_2}(x_2) = \frac{1}{2}\left(e^{-\frac{1}{3}x_2} - e^{-x_2}\right)$	
iv)	$\rho = \frac{3}{4}$	$E[\hat{r}_1] = 3.5$ $E[\hat{r}_2] = 0.5$	$f_{\underset{\sim}{x}_2}(x_2) = \frac{1}{3}\left(e^{-\frac{2}{7}x_2} - e^{-2x_2}\right)$	
v)	$\rho = 1$	$E[\hat{r}_1] = 4$ $E[\hat{r}_2] = 0$	$f_{\underset{\sim}{x}_2}(x_2) = \frac{1}{4}\, e^{-\frac{1}{4}x_2}$	

Table I : Densities for Sums of Exponential Variables

(a) Equal mean and Correlated

(b) Independent with Unequal means

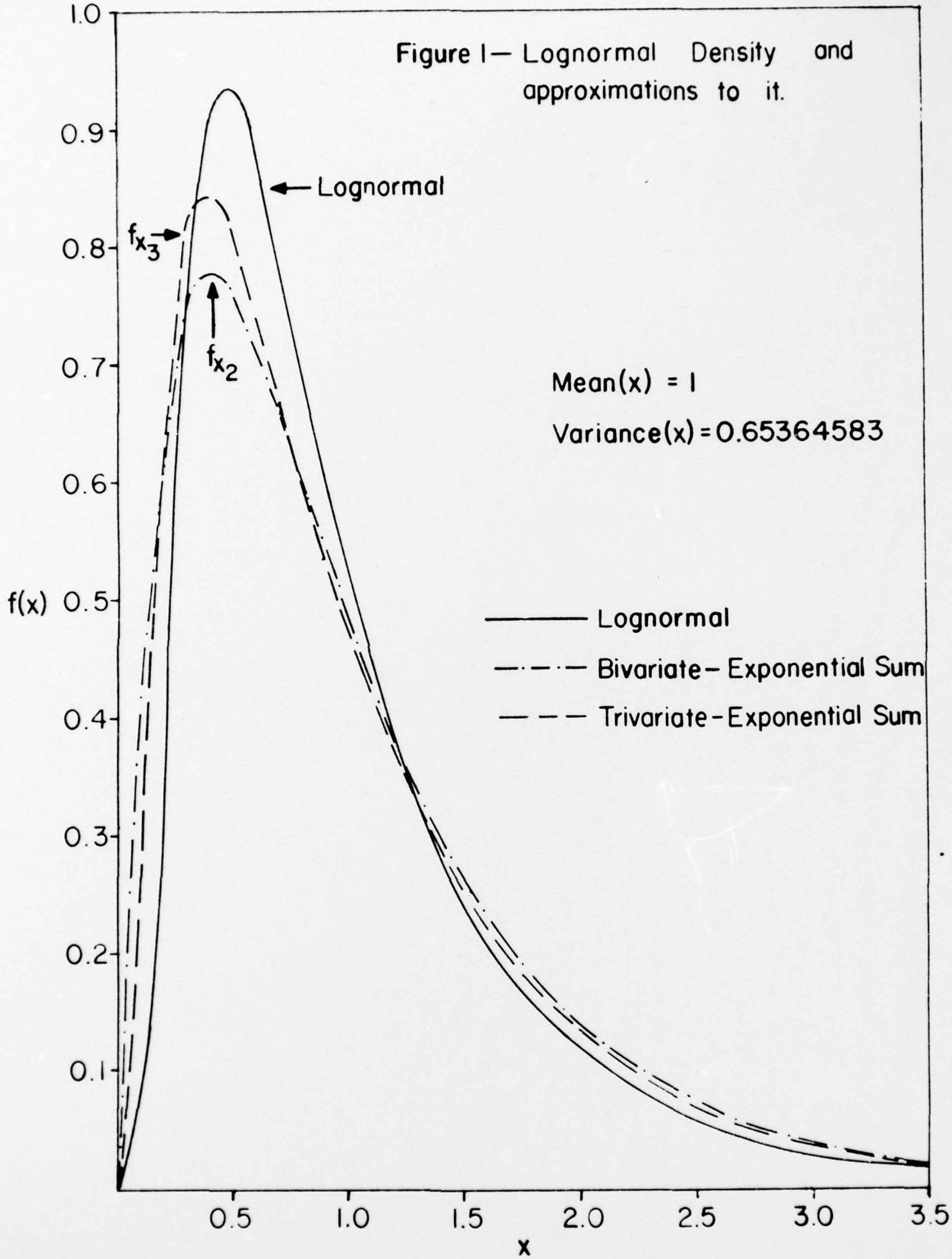


Figure I— Lognormal Density and approximations to it.
Lognormal
f_{x_3}
f_{x_2}
Mean(x) = 1
Variance(x) = 0.65364583
Lognormal
Bivariate–Exponential Sum
Trivariate–Exponential Sum
f(x)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.5
1.0
1.5
2.0
2.5
3.0
3.5
x